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Enhancement on Counting Invariant on Symmetric Virtual Biracks

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Enhancement on Counting Invariant on Symmetric Virtual Biracks

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Abstract

This thesis introduces a new enhancement for virtual birack counting invariants. We first introduce knots and other general types of knots (oriented knots, framed knots, racks, and biracks). Then we'll discuss the methods, knot invariants, mathematicians use to identify whether two knots are different. Next we'll look at knots with virtual crossings and knots with a good involution. Finally, we introduce a new symmetric enhancement for virtual birack counting invariants and provide an example.

Contents

Acknowledgments

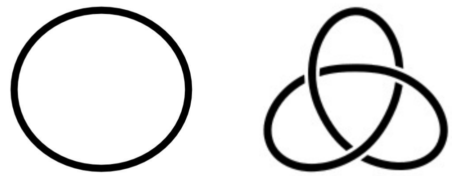
I would like to thank Professor Sam Nelson for his neverending guidance and support. I would also like to thank Professor Christopher Towse, my academic advisor, for all the support and encouragement you have given me throughout my college career. Finally, I would like to thank my family, friends, and others who have helped me get to where I am.

Chapter 1

Knots

A ***knot*** is a simple closed curve, where the curve has no loose ends, doesn't intersect itself, and has no thickness. Knots are typically studied in third-dimensional space. We are able to make a physical representation of a knot by take a piece of rope or string, contorting the rope, and then joining the two loose ends together. Another way to make a physical model, suggested by Colin Adams, is to take an extension cord, contort it, and then plugging in the plug side of the cord into the outlet end.

A few simple knots are the ***trivial knot***, which is also known as the ***unknot***, and the ***trefoil knot***. To visualize knots, mathematicians often draw knot diagrams as a projection of a knot on a plane where the overstrand is drawn unbroken and the understrand is drawn broken as it goes under an overstrand. The figures below are knot diagrams of the trivial knot and the trefoil knot:



(a) Trivial Knot (b) Trefoil Knot

Figure 1.1: Knot diagrams of the trivial knot and the trefoil knot

Another related object to knots is a ***link*** which consists of several knots that are possibly linked together. Every individual simple closed curve is a ***component*** of the link. A knot can be thought of as a link with only one component.

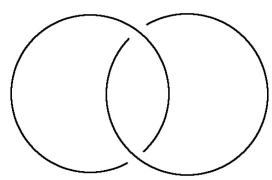
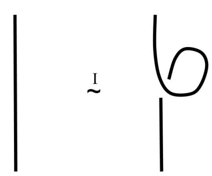


Figure 1.2: The simplest nontrivial link of two components is known as the Hopf Link.

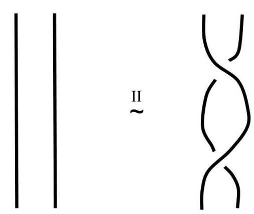
1.1 Reidemeister Moves

Two knots K_0 and K_1 are equivalent if we can rearrange knot K_0 into knot K_1 through third-dimensional space without breaking the knot and without shrinking a part of a knot into a point. If K_0 and K_1 are equivalent, we say that K_0 is ***ambient isotopic*** to K_1. In 1926, Kurt Reidemeister proved that two knot diagrams are ambient isotopic if one knot can be changed to the other by a finite sequence of three moves known as the ***Reidemeister***

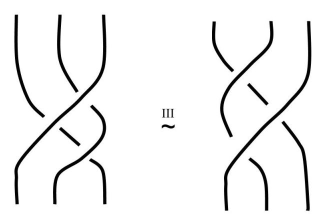
moves. The ***first Reidemeister move*** allows us to put in or take out a twist. The ***second Reidemeister move*** allows us to add or remove two crossings. The ***third Reidemeister move*** allows us to slide a strand of a knot from one side of a crossing to the other side of a crossing. With the Reidemeister moves, we can group knots by equivalence relations. The figures below show how the Reidemeister moves are applied:



(a) Reidemeister Move I



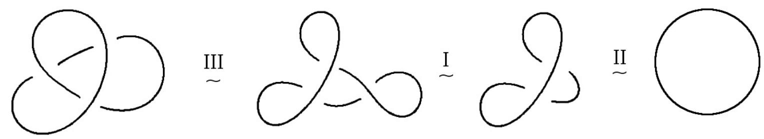
(b) Reidemeister Move II



(c) Reidemeister Move III

Figure 1.3: Knot diagrams of the three Reidemeister moves

For example, we can see that the following knot can be rearranged to become the unknot with three Reidemeister moves:



1.2 Oriented Knots and Framed Knots

For each strand in a knot, we can make the strands ***oriented*** in the same direction. They are denoted in knot diagrams by arrows. For this thesis, we will be orienting the strands downward. Since strands now have orientation, we have introduced two types of crossings: ″positive″ and ″negative″ crossings. With downward oriented strands, a crossing is positive if the understrand is directed right-to-left and a crossing is negative if the understrand is directed left-to-right. We denote positive crossings with ″$+1$″ and negative crossings with ″-1.″ The Reidemeister moves can still be applied to oriented knots as they would to unoriented knots. The ***writhe*** of a knot diagram is defined to be the sum of all crossing signs. A knot does not have a unique writhe value because the writhe can always be changed by applying the first Reidemeister move. Hence writhes are defined for knot diagrams and not for knots themselves.

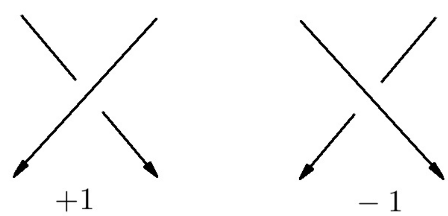


Figure 1.4: Image of a positive crossing on the left and a negative crossing on the right

Knots also have a choice of having a framing. A framed knot K is a knot where we inflate the knot like an inner tube to get a knotted torus with K as its core. A ***framing curve*** is a simple closed curve that lies on the surface of the torus and projects down onto the original knot K with an injective correspondence. We can also think of a framed knot as a torus neighborhood that is a stack of discs and points of the knot passes through the center of the discs only once and the framing curve touches the boundary of the torus and intersects the disc exactly once. Then two knots are ***frame isotopic*** if there is an ambient isotopy that takes the framing curve of one knot to the framing curve of another knot.

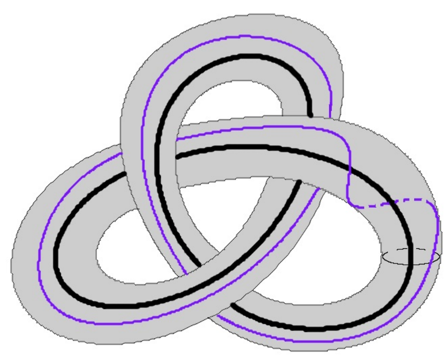


Figure 1.5: The thick black curve is the knot K and the purple curve is the framing curve

1.3 Knot Invariants

Given two knots K and K', we know that they're equivalent if we can change K into K' with the Reidemeister moves. Most of the time, it is difficult to come up with a finite sequence of Reidemeister moves that will change K into K' since it could take a large number of moves. Thus we need a different way to prove that two knots are equivalent.

A ***knot invariant*** is a function $f : \mathcal{K} \to X$ from the set of all knot diagrams to a set X such that for each Reidemeister move we have

$$f(K_{Before}) = f(K_{After})$$

where K_{Before} is the knot diagram before the move and K_{After} is the same knot diagram after the move. If f is a knot invariant, then any two diagrams related by Reidemeister moves must give the same value when f is evaluated. Knot Invariants are computable if the actual value of $f(K)$ can be determined by any diagram of K.

1.3.1 Tricoloring

An example of a computable knot invariant is the ***Fox tricoloring*** which was introduced by Ralph Fox. A ***tricoloring*** of a knot is a choice of color for each arc in the diagram from a set of three colors. A tricoloring is valid if at every crossing the strands are either all the same color or all different colors. A valid tricoloring is nontrivial if all three colors are used at every crossing.

If we want to view tricoloring as a knot invariant and we know that if we start with a valid tricoloring of a diagram K before doing a Reidemeister move, then there is a unique valid tricoloring of the diagram after the move that still keeps the entire diagram a valid tricoloring. A trivial valid way of coloring a knot diagram would be if before and after performing a

Reidemeister move, the strands use only one color. For a type II move, we also allow the two strands to be two different colors before crossing. For a type III move, we additionally allow the strands to be three different colors either before the crossing or after the crossing, as shown below.

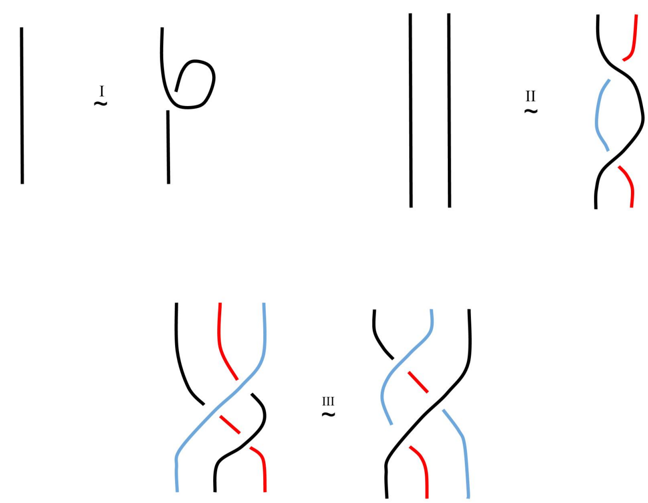


Figure 1.6: Valid tricolorings of the Reidemeister moves

Thus it is easy to see that the only valid tricoloring of the unknotted diagram is where only one color is used. On the other hand, a valid tricoloring of the trefoil uses all three colors. Therefore there are no sequence of Reidemeister moves that takes the trefoil to the unknot.

Chapter 2

Racks

Racks are framed oriented knots with the first Reidemeister move replaced with the framed type I move and they are an algebraic invariant of framed oriented knots.

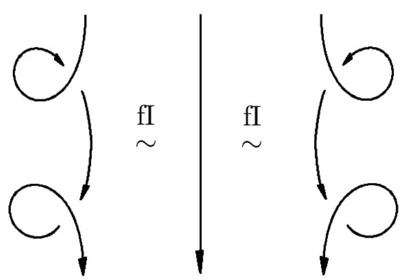


Figure 2.1: Framed Reidemeister Move I

Definition 1. *We formally define a **rack** as a set X with two binary operations $\triangleright$, $\triangleright^{-1}$: $X \times X$ that satisfies*

- $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$
- $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (x \triangleright y)$.

Then in framed isotopy, going through a kink is a bijective map $\pi : X \to X$ and is defined by $\pi(x) = x \triangleright x$ and has inverse $\pi^{-1}(x) = x \triangleright^{-1} x$. This map is known as the ***kink map***.

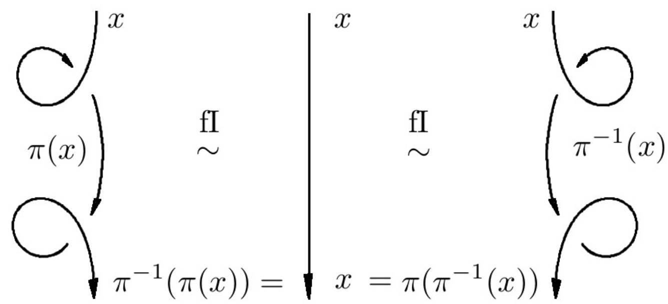


Figure 2.2: Kink Map

Then for a finite rack X, we can keep applying kinks until we get the original label again. Then for any $x \in X$, the ***rank of x*** is the smallest positive integer n such that $\pi^n(x) = x$. Then the least common multiple of these n for all elements of X is called the ***rack rank*** or ***rack characteristic*** of X. If we had a rack X which has a rack characteristic of N, then knot diagrams L and L' are equivalent by the ***N-phone cord move***:

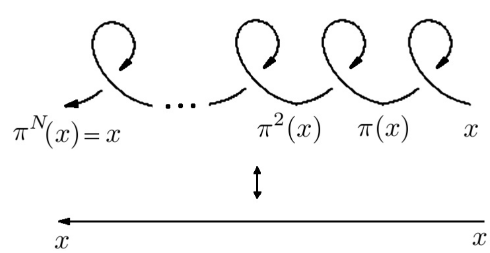


Figure 2.3: N-Phone Cord Move

2.1 Biracks

Previously, we have been putting labels on the crossings of racks. Instead we can explore the option of putting labels on the semiarcs. Thus we define a ***birack*** by the following:

Definition 2. *A **birack** is a set X with right-invertible operations $\overline{\triangleright}\,,\ \underline{\triangleright}\ : X \times X \to X$ and a bijection $\pi : X \to X$ that satisfies the following for $x, y, z \in X$:*

- $\pi(x \,\overline{\triangleright}\, x) = x \,\underline{\triangleright}\, x$ *and* $\pi(x) \,\overline{\triangleright}\, x = x \,\underline{\triangleright}\, \pi(x)$
- *The map of pairs H(x,y) =* $(y \,\overline{\triangleright}\, x, x \,\underline{\triangleright}\, y)$ *are invertible*
- *The exchange laws:*

$$(x \,\underline{\triangleright}\, y) \,\underline{\triangleright}\, (z \,\underline{\triangleright}\, y) \quad = \quad (x \,\underline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z) \tag{2.1}$$

$$(x \,\underline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\underline{\triangleright}\, y) \quad = \quad (x \,\overline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z) \tag{2.2}$$

$$(x \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\overline{\triangleright}\, y) \quad = \quad (x \,\overline{\triangleright}\, z) \,\overline{\triangleright}\, (y \,\underline{\triangleright}\, z) \tag{2.3}$$

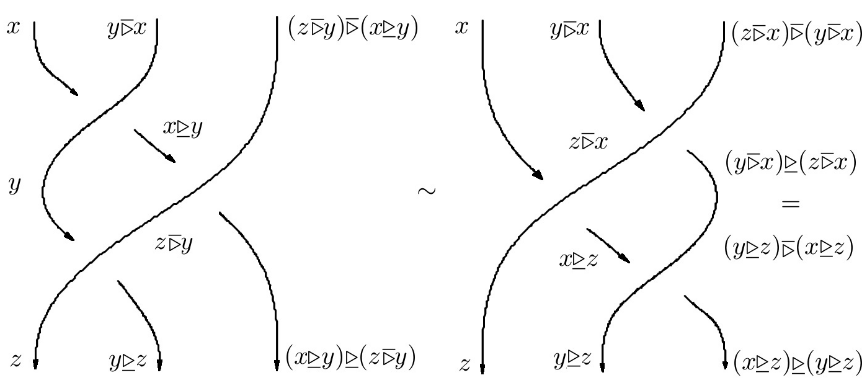


Figure 2.4: The exchange laws can be derived from the Reidemeister Move III

Similarly to racks, there exists a positive integer N for biracks such that N is the ***characteristic*** of X where $\pi^N : X \to X$ is the identity map.

2.2 Biracks Invariants

Recall that a framed knot consists of a torus with knot K as the core and a framing curve F. Then for a link L with c components, each component has its own writhe or framing curve that is independent of the other components. So a link with c components $K_1, \cdots, K_c$ has a ***framing vector*** $\vec{w} = (w_1, \cdots, w_c)$ that specifies the framing curve of each component. We call the set of framing vectors an ***integral lattice***, which we can think of as the set of all points in $\mathbb{R}^n$ with integer coordinates. For each writhe vector $\vec{w}$ there is a distinct framed version of L, which we will denoted as $L_{\vec{w}}$. Hence two links are not equivalent if they have different framing vectors ($L_{\vec{w}} \neq L_{\vec{w}'}$ if $\vec{w} \neq \vec{w}'$). Additionally if two framed knots L and L' are related by the framed Reidemeister moves and the *N-phone cord move*, then X-labelings of L and L' form a bijective correspondence. Thus we can form equivalence classes of the X-labelings of a link L.

We can think of the infinite integral lattice of the framings of L as an invariant of a unframed link L. Since we can reduce the number of framing vectors by mod N, the characteristic of link L, then we can get a canonical tiling of framing vectors that correspond to the elements of $(\mathbb{Z}_n)^c$. Then the total number of colorings on the semiarcs of the framings of L over one tile is called the ***integral birack counting invariant***, denoted by

$$\Phi_X^{\mathbb{Z}}(L) = \sum_{\vec{w} \in \mathbb{Z}_N^c} u^{|\mathcal{L}(L_{\vec{w}}, X)|}$$

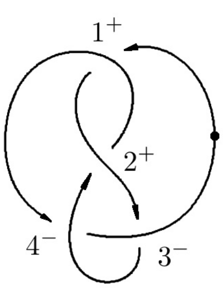
where $L_{\vec{w}}$ is a link L with framing vector $\vec{w}$ and $\mathcal{L}(L_{\vec{w}}, X)$ is the set of X-labelings of $L_{\vec{w}}$.

Chapter 3

Virtual Knots

Instead of knot diagrams, knots can also be represented as ***Gauss codes***. Given an oriented knot K, we select a basepoint on a semiarc of K and enumerate the crossings and note whether a crossing is positive or negative. While traveling in the orientation of K and starting at the basepoint, we can derive the Gauss code of K by recording the crossing number, its sign (positive or negative), and whether the strand is passing over or under until we arrive back at the basepoint.

Example 1. *Let's take a look at the knot K and basepoint as shown below:*



Then the Gauss code of K is

$$U1^{+}O2^{+}U3^{-}O4^{-}U2^{+}O1^{+}U4^{-}O3^{-}.$$

We are also capable of reconstructing a knot diagram, that's isotopic to the knot, once we are given a Gauss code by connecting the crossings as instructed.

Example 2. *If we were given this Gauss code $O1^{-}U2^{-}O3^{-}U1^{-}O2^{-}U3^{-}$, we can recreate the knot diagram as the figure below:*

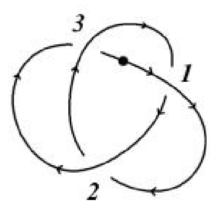
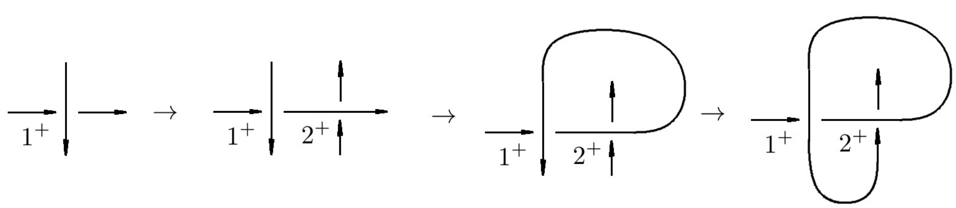


Figure 3.1: Knot diagram with Gauss code $O1^{-}U2^{-}O3^{-}U1^{-}O2^{-}U3^{-}$

However there are times when a knot diagram cannot be created.

Example 3. *If we were given this Gauss code $U1^{+}O2^{+}O1^{+}U2^{+}$, we can try to recreate the knot diagram as the figure below:*



We can't put in additional crossings because the Gauss code already includes all crossings.

We can solve this problem by introducing a new type of crossing, a ***virtual crossing***, that won't affect and are not represented in Gauss codes. Virtual crossings can be thought of as flattening the torus but actually lives into the plane. In knot diagrams, virtual crossings are denoted by a circle surrounding a crossing. There is no distinction between whether a strand is the overstrand or the understrand. Thus for the example 3 we get the knot diagram below with a virtual crossing:

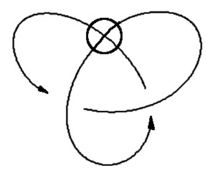


Figure 3.2: Knot diagram with Gauss Code $U1^{+}O2^{+}O1^{+}U2^{+}$ that contains a virtual crossing

Hence we have additional ***virtual Reidemeister moves:***

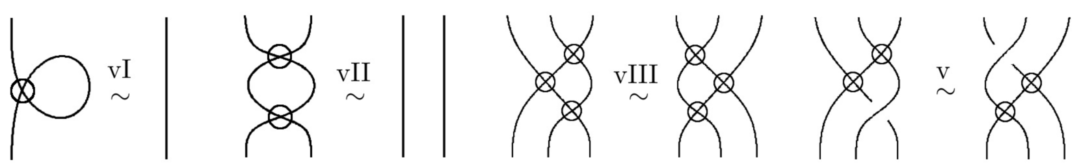


Figure 3.3: The Four Virtual Reidemeister Moves

3.1 Virtual Biracks

Biracks with classical crossings and virtual crossings are called virtual biracks.

Definition 3. *Let X be a set. A **virtual birack** structure on X consists of three binary operations* $\overline{\triangleright}$, $\underline{\triangleright}$, $\circledast : X \times X \to X$ *and a bijection* $\pi : X \times X$ *called the* *kink map that satisfies the following conditions:*

- $\pi(x) \mathbin{\bar{\triangleright}} x = x \trianglerighteq \pi(x)$ *and* $\pi(x \mathbin{\bar{\triangleright}} x) = x \trianglerighteq x$ *for all* $x \in X$
- *The maps* $\alpha_y, \beta_y, v_y : X \to X$ *defined by* $\alpha_y(x) = x \mathbin{\bar{\triangleright}} y,\ \beta_y(x) = x \trianglerighteq y,\ v_y(x) = x \circledast y$ *are bijections and the maps* $S : X \times X \to X \times X$ *and* $V : X \times X \to X \times X$ *are defined by* $S(x, y) = (y \mathbin{\bar{\triangleright}} x, x \trianglerighteq y)$ *and* $V(x, y) = (y \circledast x, x \circledast y)$ *are bijections*
- *The exchange laws are satisfied:*

$$(x \trianglerighteq y) \trianglerighteq (z \trianglerighteq y) = (x \trianglerighteq z) \trianglerighteq (y \mathbin{\bar{\triangleright}} z) \tag{3.1}$$

$$(x \trianglerighteq y) \mathbin{\bar{\triangleright}} (z \trianglerighteq y) = (x \mathbin{\bar{\triangleright}} z) \trianglerighteq (y \mathbin{\bar{\triangleright}} z) \tag{3.2}$$

$$(x \mathbin{\bar{\triangleright}} y) \mathbin{\bar{\triangleright}} (z \mathbin{\bar{\triangleright}} y) = (x \mathbin{\bar{\triangleright}} z) \mathbin{\bar{\triangleright}} (y \trianglerighteq z) \tag{3.3}$$

$$(x \circledast y) \circledast (z \circledast y) = (x \circledast z) \circledast (y \circledast z) \tag{3.4}$$

$$(x \mathbin{\bar{\triangleright}} y) \circledast (z \circledast y) = (x \circledast z) \mathbin{\bar{\triangleright}} (y \circledast z) \tag{3.5}$$

$$(x \circledast y) \circledast (z \mathbin{\bar{\triangleright}} y) = (x \circledast z) \circledast (y \trianglerighteq z) \tag{3.6}$$

$$(x \trianglerighteq y) \circledast (z \circledast y) = (x \circledast z) \trianglerighteq (y \circledast z) \tag{3.7}$$

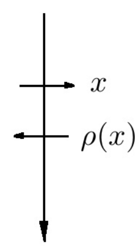
Thus Axiom 2 implies that operations $\trianglerighteq$, $\bar{\triangleright}$, and $\circledast$ are right-invertible, so we denote the right inverse operations as $\trianglerighteq^{-1}$, $\bar{\triangleright}^{-1}$, and $\circledast^{-1}$. A virtual birack is ***involutory*** if $*^{-1} = *$ for $* \in \{\bar{\triangleright}, \trianglerighteq, \circledast\}$.

Definition 4. *Let* X *be a virtual birack. An involution* $\rho : X \times X$ *is a* ***good involution*** *if for all* $x, y \in X$*,*

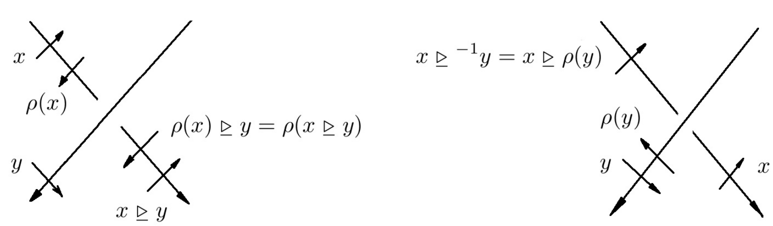
$$\rho(x) * y = \rho(x * y) \quad \textit{and} \quad x * \rho(y) = x *^{-1} y$$

where $* \in \{ \trianglerighteq, \overline{\triangleright}, \circledast \}$. *A virtual birack with a choice of good involution is a* ***symmetric virtual birack***.

We can represent $\rho(x)$ in knot diagrams where the label for semiarcs is an arrow that is perpendicular to the semiarc, then $\rho(x)$ is pointing in the opposite direction of x .



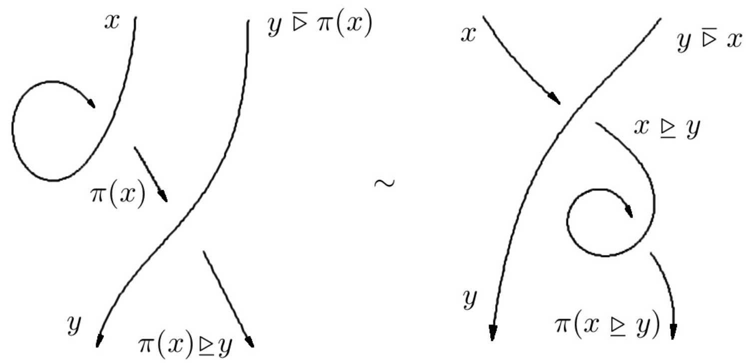
Additional good involution conditions are created when $x * y$ is applied at a crossing and pushed through the direction of the strand, as shown below.



Given a virtual birack, we want to know which involutions $\rho : X \times X$ is a good involution. Proposition 3.4 from [7] states that the identity map of X is a good involution.

Proposition 3.1.1. *If X is a birack, then the kink map* $\pi : X \to X$ *is a good involution if and only if X is involutory.*

Proof. We must show that the kink map satisfies the good involution conditions, $\rho(x) * y = \rho(x * y)$ and $x * \rho(y) = x *^{-1} y$.



From the figure above, we know that

$$y \,\overline{\triangleright}\, x = x \,\overline{\triangleright}\, \pi(x) \quad \textit{and} \quad \pi(x \,\underline{\triangleright}\, y) = \pi(x) \,\underline{\triangleright}\, y.$$

If we replaced the crossings of the figure above with negative crossings and virtual crossings, we can also conclude that

$$y \,\underline{\triangleright}\, x = y \,\underline{\triangleright}\, \pi(x) \quad \textit{and} \quad \pi(x \,\overline{\triangleright}\, y) = \pi(x) \,\overline{\triangleright}\, y$$

and

$$y \circledast x = y \circledast \pi(x) \quad \textit{and} \quad \pi(x \circledast y) = \pi(x) \circledast y$$

Then if X is involuntory, we have

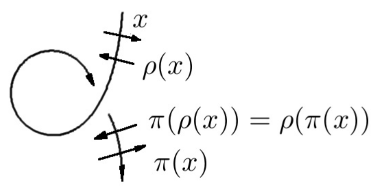
$$x \,\underline{\triangleright}\, \pi(x) = x \,\underline{\triangleright}\, y = x \,\underline{\triangleright}^{-1} y \quad \textit{and} \quad y \,\overline{\triangleright}\, \pi(x) = y \,\overline{\triangleright}\, x = y \,\overline{\triangleright}^{-1} x$$

and π is a good involution. Conversely, if X is involutory then π satisfies the conditions for being a good involution.

□

Lemma 3.1.2. *We can make an observation that if X is a virtual birack and ρ : $X \times X$ is a good involution, then $\pi\rho = \rho\pi$.*

Proof.



□

Chapter 4

Enhancements of Symmetric Virtual Biracks

Recall that the total number of colorings of framings of L over one tile is called the integral birack counting invariant, denoted by

$$\Phi_X^{\mathbb{Z}}(L) = \sum_{\vec{w} \in \mathbb{Z}_N^c} u^{|\mathcal{L}(L_{\vec{w}}, X)|}$$

where $L_{\vec{w}}$ is a link L with framing vector $\vec{w}$ and $\mathcal{L}(L_{\vec{w}}, X)$ is the set of X-labelings of $L_{\vec{w}}$. To extend this to the virtual birack case, we ignore virtual crossings when we determine colorings. Thus the ***integral virtual birack counting invariant*** is denoted by

$$\Phi_X^{\mathbb{Z}}(L) = \sum_{\vec{w} \in \mathbb{Z}_N^c} u^{|\mathcal{L}(L_{\vec{w}}, X)|}.$$

Next, let's look at virtual biracks with the good involution ρ. Then we say that two X-labelings of a link L are ***ρ-equivalent*** if we can get one X-labeling to become the other by applying ρ to a subset of the semiarc labels. In other words, if two labelings are ρ-equivalent and if for every semiarc that's labeled x in one labeling, the corresponding semiarc in the other labeling must be x or $\rho(x)$. So there is an equivalence relation between X-labelings that are ρ-equivalent that partitions the set of labelings into disjoint subsets. We denote these disjoint subsets (also known as quotient sets) by $\mathcal{L}(L_{\vec{w}}, X)/\rho$.

Definition 5. *Let X be a virtual birack with good involution ρ. Then the* ***symmetric enhancement*** *(an enhancement of a knot invariant is another invariant that can recover the original counting invariant) of the virtual birack counting invariant is*

$$\Phi_X^\rho(L) = \sum_{\vec{w}\in\mathbb{Z}_N^c} \left(\sum_{x\in\mathcal{L}(L_{\vec{w}},X)/\rho} u^{|x|} \right).$$

If ρ has no fixed points, then for every X-labeling of a diagram L there is exactly one other ρ-equivalent X-labeling which is acquired by applying ρ to every label. Then the enhanced invariant is equivalent to the unenhanced variant by

$$\Phi_X^\rho(L) = \frac{1}{2}\Phi_X^{\mathbb{Z}}(L)u^2.$$

If $\rho = \mathrm{Id}_X$, then we have

$$\Phi_X^\rho(L) = \Phi_X^{\mathbb{Z}}(L)u.$$

However if $\rho \neq \mathrm{Id}$ has fixed points, the equivalence classes by the unenhanced invariant can have different sizes. Hence the enhanced invariant can contain more information about L than the unenhanced invariant.

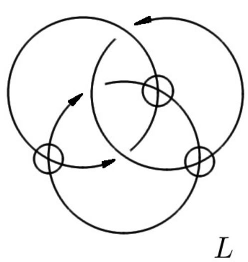
Example 4. *Let X be the virtual birack X of characteristic $N = 2$ with operation matrix*

$$\left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 3 & 2 & 2 & 3 & 3 & 3 \\ 3 & 2 & 2 & 2 & 3 & 3 & 2 & 2 & 2 \end{array}\right]$$

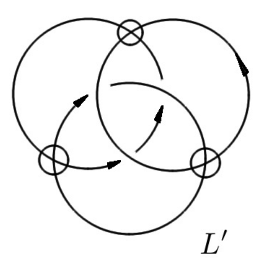
where

$$x_k = \begin{cases} x_i \trianglerighteq x_j & 1 \leq i, j \leq n \\ x_i \,\overline{\triangleright}\, x_j & n+1 \leq i, j \leq 2n \\ x_i \circledast x_j & 2n+1 \leq i, j \leq 3n. \end{cases}$$

and let $\rho : X \to X$ be the permutation $\rho = (34)$. Then our `python` *computations reveal that the two virtual links L and L' both have a counting invariant value $\Phi_X^{\mathbb{Z}}(L) = \Phi_X^{\mathbb{Z}}(L') = 120$, but the links are distinguishable by their $\Phi_X^{\rho}(L)$ values:*



(a) $\Phi_X^{\rho}(L) =$
$u^8 + 12u^4 + 32u^2$



(b)
$\Phi_X^{\rho}(L') = u^8+12u^4+64u$

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